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On the convergence of the rational interpolation approximant of Carathéodory functions

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Abstract

Let $\alpha = \{z_{n,m}\}_{m=1}^{n}$ with $|z_{n,m}| < 1$, $n = 1, 2, \ldots$, be an arbitrary sequence of complex numbers. We study the convergence of the rational function p_{n-1}/q_n which interpolates the Carathéodory function $F(z) = 1/(2\pi)\int_0^{2\pi} u/(u-z)\,\mathrm{d}\mu(t)$, $u = \mathrm{e}^{\mathrm{i}t}$, at $\{z_{n,m}, 1/\bar{z}_{n,m}\}_{m=1}^{n}$, where $\mathrm{d}\mu$ is a finite positive measure on the unit circle. We also consider the relations between p_{n-1}/q_n and the orthogonal polynomials with respect to varying measures.

Keywords: Orthogonal polynomials; Rational interpolation

1. Introduction and notation

Let $\mathrm{d}\mu$ be a finite positive Borel measure on the unit circle $\Gamma := \{z \in \mathbb{C}: |z| = 1\}$ whose support consists of infinitely many points. Let

$$F(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{u}{u-z}\,\mathrm{d}\mu(t), \quad u = \mathrm{e}^{\mathrm{i}t}.$$

This is a Carathéodory function and it is analytic in $|z| \neq 1$ with $\operatorname{Re}\hat{F}(0) > 0$. Consider an arbitrary table $\alpha = \{z_{n,k}\}$, where $k = 1, \ldots, n$, $n \in \mathbb{N}$ and $|z_{n,k}| < 1$. For each n, we define the orthonormal polynomials $\phi_{n,m}(z) = \kappa_{n,m} z^m + \cdots$, $\kappa_{n,m} > 0$, with respect to $\mathrm{d}\mu/|w_n(u)|^2$, i.e., polynomials satisfying

$$\frac{1}{2\pi}\int_0^{2\pi} u^{-j}\phi_{n,m}(u)\frac{\mathrm{d}\mu(t)}{|w_n(u)|^2} = 0, \quad j = 0, 1, \ldots, m-1,$$

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and

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi_{n,m}(u)|^2\frac{\mathrm{d}\mu(t)}{|w_n(u)|^2}=1,\quad u=\mathrm{e}^{\mathrm{i}t},$$

where $w_n(u)=\prod_{k=1}^n(1-\bar z_{n,k}u)$. The sequences $\{\phi_{n,m}\}_{m=0}^\infty$, $n=1,\dots,$ are called *orthonormal polynomials* with respect to varying measures $\mathrm{d}\mu/|w_n(u)|^2$. The purpose of this paper is to study the $(n-1,\ n)$ rational function which interpolates F at the points $\{z_{n,i},\ 1/\bar z_{n,i}\}_{i=1}^n$, $n\in\mathbb{N}$. It is easy to verify that there is a rational function R_n of the form p_{n-1}/q_n (p_{n-1} and q_n are polynomials in z with $\deg p_{n-1}\leqslant n-1$, $\deg q_n\leqslant n$) which interpolates F at the points $\{z_{n,i}\}_{i=1}^n$ of the nth row of α and $\{1/\bar z_{n,i}\}_{i=1}^n$ (cf. Section 2). We find out that the denominator for R_n is closely connected to $\phi_{n,n}(z)$. It is known that $\phi_{n,n}(z)$ is closely related to problems concerned with orthogonal rational functions (see, for example, [1,3,9]). The asymptotic properties of $\phi_{n,n}$ which have been investigated extensively by some authors (see, for example, [6,8]), enable us to discuss the convergence of R_n.

The rational interpolation with free poles of functions of Markov–Stieltjes type has been studied by several authors (cf. [4,5,7]). More recently, the Newton-type rational interpolation for Carathéodory functions at $\{\alpha_i\}_{i=1}^\infty$, with $|\alpha_i|<1$, has been considered (cf. [1,2]). We generalize their results into an arbitrary table α.

The main results are stated in Section 2, and their proofs are given in Section 3.

2. Main results

Let

$$\psi_n(z):=\frac{1}{2\pi}\int_0^{2\pi}[\xi_n(z)-\xi_n(u)]D(u,\ z)\ \mathrm{d}\mu(t),\quad u=\mathrm{e}^{\mathrm{i}t},$$

where $\xi_n(z):=\phi_{n,n}(z)/w_n(z)$ and $D(u,\ z)=u/(u-z)$; then it is easy to see that $w_n(z)\psi_n(z)$ is a polynomial with degree less than or equal to $n-1$ since $\xi_n(u)-\xi_n(z)$ vanishes for $u=z$.

We now observe the connection between the $(n-1,\ n)$ rational interpolation and $\phi_{n,n}(z)$.

Theorem 1. *Let $p_{n-1}(z):=\psi_n(z)w_n(z)$. Then $p_{n-1}/\phi_{n,n}$ interpolates $F(z)$ at $\{z_{n,k},\ 1/\bar z_{n,k}\}_{k=1}^n$ in the following sense:*

$$\phi_{n,n}(z_{n,i})F(z_{n,i})-p_{n-1}(z_{n,i})=0 \tag{1}$$

and

$$\phi_{n,n}\left(\frac{1}{\bar z_{n,i}}\right)F\left(\frac{1}{\bar z_{n,i}}\right)-p_{n-1}\left(\frac{1}{\bar z_{n,i}}\right)=0, \tag{2}$$

where $i=1,\ 2,\dots,n$. Furthermore, $u=\mathrm{e}^{\mathrm{i}t}$,

$$\phi_{n,n}(z)F(z)-p_{n-1}(z)=\frac{w_n(z)v_n(z)}{2\pi}\int_0^{2\pi}\frac{u}{(u-z)}\,\frac{\phi_{n,n}(u)}{w_n(u)v_n(u)}\ \mathrm{d}\mu(t), \tag{3}$$

where $v_n(z):=\prod_{k=1}^n(z-z_{n,k})$.

Remark 2. From (3), if $\phi_{n,n}(z_{n,k})=0$ for some k, then $p_{n-1}(z_{n,k})=0$. Thus we can define

$$\frac{p_{n-1}(z_{n,k})}{\phi_{n,n}(z_{n,k})}=\lim_{z\to z_{n,k}}\frac{p_{n-1}(z)}{\phi_{n,n}(z)}.$$

Also all the zeros of $\phi_{n,n}$ lie in $|z|<1$; so from (2) we have

$$F\left(\frac{1}{\bar{z}_{n,i}}\right)=\frac{p_{n-1}(1/\bar{z}_{n,i})}{\phi_{n,n}(1/\bar{z}_{n,i})},\quad i=1,\ldots,n.$$

For the convergence theorem, we have the following theorem.

Theorem 3. *If* $\lim_{n\to\infty}\Sigma_{i=1}^{n}(1-|z_{n,i}|)=\infty$, *then*

$$\lim_{n\to\infty}\frac{p_{n-1}(z)}{\phi_{n,n}(z)}=F(z),$$

locally uniformly in $|z|>1$.

If we have more information from α, then we can get the rate of convergence. Let $\Phi(z)$ be a function defined in $|z|>1$ and $|\Phi(z)|<1$ in $|z|>1$. α is called *uniformly distributed* with respect to the function $\Phi(z)$ if

$$\lim_{n\to\infty}\left(\frac{w_n(z)}{v_n(z)}\right)^{1/n}=\Phi(z), \tag{4}$$

locally uniformly in $|z|>1$. For the rate of convergence, we have the following theorem.

Theorem 4. *Let* α *be uniformly distributed with respect to the function* $\Phi(z)$. *Then*,

$$\lim_{n\to\infty}\left[F(z)-\frac{p_{n-1}(z)}{\phi_{n,n}(z)}\right]^{1/n}=\Phi(z), \tag{5}$$

locally uniformly in $|z|>1$.

Remark 5. It is well known that if α is uniformly distributed with respect to the function $\Phi(z)$, then $\lim_{n\to\infty}\Sigma_{i=1}^{n}(1-|z_{n,i}|)=\infty$ (cf. [10]).

3. Proofs of theorems

We need to introduce some notation. The $*$-transform $P_n^*(z)$ of a polynomial $P_n(z)$ of degree n is defined as $P_n^*(z)=z^n\overline{P_n(1/\bar{z})}$. One can check that $|P_n^*(z)|=|P_n(z)|$ for $z\in\Gamma$. Before we give the proofs, we isolate some lemmas on which our proofs are based.

Lemma 6. *For any polynomial q_n with degree less than or equal to n, we have, for $|z| \neq 1$,*

$$\frac{\psi_n(z)}{T_n(z)} = \frac{1}{2\pi}\int_0^{2\pi} D(u, z)\left[\frac{\xi_n(z)}{T_n(z)} - \frac{\xi_n(u)}{T_n(u)}\right] \mathrm{d}\mu(t),$$

where $T_n(z) := v_n(z)/q_n(z)$ and $u = \mathrm{e}^{\mathrm{i}t}$.

Proof. We only have to check that, $u = \mathrm{e}^{\mathrm{i}t}$,

$$\frac{1}{2\pi}\int_0^{2\pi} D(u, z)\left[1 - \frac{T_n(z)}{T_n(u)}\right]\xi_n(u)\, \mathrm{d}\mu(t) = 0.$$

Since the term in square brackets vanishes for $z = u$, then, $u = \mathrm{e}^{\mathrm{i}t}$,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} D(u, z)\left[1 - \frac{T_n(z)}{T_n(u)}\right]\xi_n(u)\, \mathrm{d}\mu(t) \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} D(u, z)\left[1 - \frac{T_n(z)q_n(u)}{v_n(u)}\right]\frac{\phi_{n,n}(u)}{w_n(u)}\, \mathrm{d}\mu(t) \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \frac{uQ_{n-1}(u)}{v_n(u)}\,\frac{\phi_{n,n}(u)}{w_n(u)}\, \mathrm{d}\mu(t) \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \bar{u}^{n-1}Q_{n-1}(u)\frac{\phi_{n,n}(u)}{|w_n(u)|^2}\, \mathrm{d}\mu(t) \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \overline{Q_{n-1}^*(u)}\phi_{n,n}(u)\frac{\mathrm{d}\mu(t)}{|w_n(u)|^2} = 0,
\end{aligned}$$

where Q_{n-1} is a polynomial with degree less than or equal to $n-1$. □

The following weak-star convergence result plays an important role in the study of convergence of R_n.

Lemma 7 (cf. [6,8]). *If* $\lim_{n\to\infty}\sum_{i=1}^{n}(1 - |z_{n,i}|) = \infty$, *then*

$$\left|\frac{w_n(u)}{\phi_{n,n}(u)}\right|^2 |\mathrm{d}u| \to \mathrm{d}\mu, \quad n \to \infty,$$

in the weak-star topology.

Proof of Theorem 1. From Lemma 6, setting $q_n(z) = 1$, we have (3). Note that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{u\phi_{n,n}(u)}{w_n(u)v_n(u)}\, \mathrm{d}\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\bar{u}^{n-1}\phi_{n,n}(u)}{|w_n(u)|^2}\, \mathrm{d}\mu(t) = 0, \quad u = \mathrm{e}^{\mathrm{i}t},$$

which implies that $u\phi_{n,n}(u)/w_n(u)v_n(u) \in L_1(\mu)$. Therefore, $u = e^{it}$,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{u-z} \frac{u\phi_{n,n}(u)}{w_n(u)v_n(u)} \, d\mu(t)$$

is analytic in $|z| \neq 1$ as a Cauchy–Stieltjes integral. From (3), we have (1) and (2). □

Proof of Theorem 3. From the definition of $\psi_n(z)$, we have, for $|z| \neq 1$,

$$F(z) - \frac{p_{n-1}(z)}{\phi_{n,n}(z)} = \frac{w_n(z)}{2\pi\phi_{n,n}(z)} \int_0^{2\pi} \frac{u}{u-z} \frac{\phi_{n,n}(u)}{w_n(u)} \, d\mu(t), \quad u = e^{it}. \tag{6}$$

First note that, $u = e^{it}$,

$$\begin{aligned} &\left| \frac{1}{2\pi}\int_0^{2\pi} \frac{u}{(u-z)} \frac{\phi_{n,n}(u)}{w_n(u)} \, d\mu(t) \right| \\ &\leqslant \frac{1}{2\pi(|z|-1)} \int_0^{2\pi} \frac{|\phi_{n,n}(u)|}{|w_n(u)|} \, d\mu(t) \\ &\leqslant \frac{1}{|z|-1} \left(\frac{1}{2\pi}\int_0^{2\pi} d\mu(t) \right)^{1/2} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{|\phi_{n,n}(u)|^2}{|w_n(u)|^2} \, d\mu(t) \right)^{1/2} \\ &\leqslant \frac{1}{|z|-1} \left(\frac{1}{2\pi}\int_0^{2\pi} d\mu(t) \right)^{1/2}. \end{aligned} \tag{7}$$

Next, we claim that

$$\lim_{n\to\infty} \frac{w_n(z)}{\phi_{n,n}(z)} = 0, \tag{8}$$

locally uniformly in $|z| > 1$. We only need to prove that

$$\lim_{n\to\infty} \frac{v_n(z)}{\phi^*_{n,n}(z)} = 0,$$

locally uniformly in $|z| < 1$.

Since all zeros of $\phi^*_{n,n}(z)$ lie in $|z| > 1$, $w_n(z)/\phi^*_{n,n}(z)$ is analytic in $|z| \leqslant 1$. Then, for $u = e^{it}$,

$$\begin{aligned} \left| \frac{w_n(z)}{\phi^*_{n,n}(z)} \right| &= \left| \frac{1}{2\pi}\int_0^{2\pi} \frac{u}{u-z} \frac{w_n(z)}{\phi^*_{n,n}(u)} \, dt \right| \leqslant \frac{1}{1-|z|} \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{w_n(u)}{\phi^*_{n,n}(u)} \right| dt \\ &\leqslant \frac{1}{1-|z|} \left(\frac{1}{2\pi}\int_0^{2\pi} \left| \frac{w_n(u)}{\phi_{n,n}(u)} \right|^2 dt \right)^{1/2}. \end{aligned} \tag{9}$$

From Lemma 7, we know that $w_n(z)/\phi^*_{n,n}(z)$ is bounded locally uniformly in $|z| < 1$. Note that $v_n(z)/w_n(z)$ is essentially a Blaschke product, thus

$$\lim_{n\to\infty} \frac{v_n(z)}{w_n(z)} = 0,$$

locally uniformly in $|z| < 1$, since $\lim_{n\to\infty}\Sigma_{m=1}^{n}(1 - |z_{n,m}|) = \infty$ (cf. [10]). This implies the claim. Using (6)–(8) completes the proof of the theorem. □

Proof of Theorem 4. From (9), we know that $v_n(z)/\phi_{n,n}(z)$ is bounded locally uniformly in $|z| > 1$. Thus we have, from (4),

$$\lim_{n\to\infty}\left[\frac{w_n(z)}{\phi_{n,n}(z)}\right]^{1/n} = \lim_{n\to\infty}\left[\frac{w_n(z)}{v_n(z)}\,\frac{v_n(z)}{\phi_{n,n}(z)}\right]^{1/n} = \Phi(z), \tag{10}$$

locally uniformly in $|z| > 1$. Using (6), together with (7) and (10), yields the theorem. □

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